Discrete Structures, Fall 2023, Homework 9

You must write the solutions to these problems legibly on your own paper, with the problems in sequential order, and with all sheets stapled together.

1. Prove the following statement using an *element proof*:

   For any sets A, B, C, and D, if $C \subseteq (D \cup A)$ and $B \subseteq D^c$, then $C \cap B \subseteq A$.

2. Prove the following statement using an *element proof*:

   For any sets A, B, and C, if $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

   Hint: You will need to divide the proof into two cases at some point, like in practice problem 2 below.

SET PRACTICE PROBLEMS: (*These are not part of the homework; solutions are on the next page.*)

1. Prove the following statement using an *element proof*:

   For any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

2. Prove the following statement using an *element proof*:

   For any sets A, B, and C, if $A \subseteq B$, then $(A \cup C) \subseteq (B \cup C)$.

3. Prove the following statement using an *element proof*:

   For any sets A, B, C, D, and E, if $A \subseteq (B \cup C)^c$ and $D \subseteq E$, then $(A \cap D) \subseteq (E - B)$.

SOLUTIONS TO PRACTICE PROBLEMS:

1. Prove the following statement using an *element proof*:

   For any sets A, B, and C, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

   Proof:

   Let A, B, and C be arbitrary sets.

   Assume $A \subseteq B$ and $B \subseteq C$.

   [*Note: We are trying to prove that $A \subseteq C$, so let's assume x is an arbitrary element in the left side of the subset (A) and show x must be in the right side (C).*]

   Let x be an arbitrary element in U. Assume $x \in A$.

   [*Note: The line above could also be written as "Let x be an arbitrary element in A."*]

   $x \in A$ (from above)

   $x \in B$ (because $A \subseteq B$ and $x \in A$)

   $x \in C$ (because $B \subseteq C$ and $x \in B$)

   Because we assumed x was an arbitrary element in A and we showed $x \in C$, we can conclude (by the definition of subset) that $A \subseteq C$.

2. Prove the following statement using an *element proof*:

   For any sets A, B, and C, if $A \subseteq B$, then $(A \cup C) \subseteq (B \cup C)$.

   Proof:

   Let A, B, and C be arbitrary sets.

   Assume $A \subseteq B$.

   [*Note: We are trying to prove that $(A \cup C) \subseteq (B \cup C)$, so let's assume x is an arbitrary element in the left side of the subset ($A \cup C$) and show x must be in the right side ($B \cup C$).*]

   Let x be an arbitrary element in U. Assume $x \in A \cup C$.

   [*Note: The line above could also be written as "Let x be an arbitrary element in $A \cup C$."*]

   $x \in A \cup C$ (from above)

   $x \in A \vee x \in C$ (def of union)

   [*Note: Because we don't know if $x \in A$ or if $x \in C$, we will split the proof into cases, one case where we assume $x \in A$ and one where we assume $x \in C$. We will make sure both cases lead to the same conclusion, that $x \in B \cup C$. And then we will use the rule of dilemma/proof by division into cases to conclude that whichever side of the "or" statement above is true ($x \in A \vee x \in C$), since both sides lead to the same conclusion (that $x \in B \cup C$), that the conclusion must be true in general.*]

   Case 1: Assume $x \in A$.

   $x \in B$ (because $x \in A$ and $A \subseteq B$)

   $x \in B \vee x \in C$ (disjunctive addition)

   $x \in B \cup C$ (def of union)

Case 2: Assume $x \in C$.

$x \in B \vee x \in C$ (disjunctive addition)

$x \in B \cup C$ (def of union)

Since both cases above lead to the same conclusion, we can conclude that $x \in B \cup C$.

Because we assumed x was an arbitrary element in $A \cup C$ and we showed $x \in B \cup C$, we can conclude (by the definition of subset) that $A \cup C \subseteq B \cup C$.

3. Prove the following statement using an *element proof*:

For any sets A, B, C, D, and E, if $A \subseteq (B \cup C)^c$ and $D \subseteq E$, then $(A \cap D) \subseteq (E - B)$.

Proof:

Let A, B, C, D, and E be arbitrary sets.

Assume $A \subseteq (B \cup C)^c$ and $D \subseteq E$.

[*Note: We are trying to prove that $(A \cap D) \subseteq (E - B)$, so let's assume x is an arbitrary element in the left side of the subset $(A \cap D)$ and show x must be in the right side $(E - B)$.*]

Let x be an arbitrary element in U. Assume $x \in A \cap D$.

[*Note: The line above could also be written as "Let x be an arbitrary element in $A \cap D$."*]

$x \in A \cap D$ (from above)

$x \in A \wedge x \in D$ (def of intersection)

$x \in A$ (conjunctive simplification)

$x \in D$ (conjunctive simplification)

$x \in (B \cup C)^c$ (because $x \in A$ and $A \subseteq (B \cup C)^c$)

$\sim(x \in (B \cup C))$ (def of complement)

$\sim(x \in B \vee x \in C)$ (def of union)

$\sim(x \in B) \wedge \sim(x \in C)$ (deMorgan's law)

$\sim(x \in B)$ (conjunctive simplification)

$x \in B^c$ (def of complement)

[*Note: We also could have turned $x \in (B \cup C)^c$ into $x \in (B^c \cap C^c)$ through the set version of deMorgan's laws, and proceeded from there. We would still want to get $x \in B^c$.*]

$x \in E$ (because $x \in D$ and $D \subseteq E$)

$x \in E \wedge x \in B^c$ (conjunctive addition)

$x \in E \cap B^c$ (def of intersection)

$x \in E - B$ (def of set difference)

Because we assumed x was an arbitrary element in $A \cap D$ and we showed $x \in B - E$, we can conclude (by the definition of subset) that $A \cap D \subseteq E - B$.